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Abstract: In this paper, we investigate Partial Observability Normal Forms (PONF) of nonlinear dynamical systems. Necessary and sufficient conditions for the existence of a diffeomorphism bringing the original nonlinear system into a PONF are established. This enables us to estimate a part of state of a nonlinear dynamical system. A concrete example (SIR epidemic model) is provided to illustrate the feasibility of the proposed results.

1 Introduction

Since the last four decades, many research activities have been developed to deal with the problem of state estimation of nonlinear dynamical systems. Several nonlinear state estimation methods have been performed to improve accuracy and performances of the control system design. Generally, we distinguish two approaches for nonlinear observer design. The first one is to design observer directly for the nonlinear systems which however highly depends on the studied system and there does not exist a uniform way to study general nonlinear systems. The second approach is based on some nonlinear transformations, using Lie algebra, to bring the original system into canonical observability normal form, from which the design of state observers is performed by using existing observer techniques in the new coordinates. The literature is vast about this second approach since the pioneer works of ([1, 2]) for single output systems and [3] for the case of multi outputs (see also [4–18]).

All the above-mentioned papers are dedicated to the full order case (i.e. the observer and the original system have the same dimension) and few works have been dedicated to partial observation which makes sense , in practice, when only a part or a function of states are required. Among the papers dedicated to this issue, let us quote the work of [19] on Z-observability or in [20–22] where the authors proposed nonlinear observer based on particular canonical forms.

This paper proposes a PONF for partially observable nonlinear systems. In the new coordinates, a simple Luenberger observer is used to estimate a part of state of the studied system. Necessary and sufficient conditions are established to transform the original nonlinear system into the PONF.

This paper is organized as follows. Section 2 recalls Z-observability. In Section 3, PONF is presented. Necessary and sufficient conditions are deduced in Section 4 to bring the original nonlinear system into the PONF. An extension to nonlinear systems with inputs is presented in section 5. Section 6 generates the results by applying another diffeomorphism on the output, in which a concrete example (SIR epidemic model) is presented in order to highlight our results.

2 Z-observability

Let us consider the following nonlinear dynamical system:

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}$ is the output, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are analytic. Contrary to the classical observability analysis as in [2], where the full state vector is estimated, this paper considers the observability of the following variables

$$z = l\left(x\right) \tag{2}$$

where $z \in \mathbb{R}^p$. This problem was firstly studied in [19], and is named as Z-observability.

Definition 1 *(Z-observability) $z = l\left(x\right)$ is said to be Z-observable with respect to system (1), if for any two trajectories, $x^i(t)$, $1 \leq i \leq 2$, in $U \subset \mathbb{R}^n$ defined on a same interval $[t_0, t_1]$, the equality*

$$h(x^1(t)) = h(x^2(t)), \text{ a.e. in } [t_0, t_1]$$

implies

$$Z(x^1(t)) = Z(x^2(t)), \text{ a.e. in } [t_0, t_1]$$

If for any trajectory $x(t)$ in U there always exists an open set $U_1 \subset U$ so that z is Z-observable in U_1, then z is said to be locally Z-observable in U.

The above definition of Z-observability can be interpreted in an algebraic way, which is linked to the classical definition of algebraical observability in [23]. In this work we will adopt the following definition.

Definition 2 *$z = l\left(x\right)$ is said to be Z-observable with respect to system (1), if it can be expressed as functions of the output and its derivatives, i.e.*

$$z = l\left(x\right) = \tilde{l}\left(y, \dot{y}, \cdots, y^{(i)}, \cdots\right)$$

In the following, by assuming that z is Z-observable, we are going to propose a universal approach to estimate z for system (1). This method is based on transforming nonlinear system (1) into a so-called partial observability normal form, from which a reduced order observer can be easily designed.

3 Partial observability normal form

This paper considers only the non trivial case of Z-observability, i.e. it is assumed that for dynamical system (1) we have rank$\left\{dh, dL_f h, \cdots, dL_f^k h, \cdots\right\} = r < n$.

Let consider the following partial observability normal form

$$\left\{\begin{array}{l} \dot{\xi} = A\xi + \beta(y) \\ \dot{\zeta} = \eta(\xi,\zeta) \\ y = C\xi \end{array}\right. \tag{3}$$

where $\xi \in \mathbb{R}^r$, $\zeta \in \mathbb{R}^{n-r}$, $y \in \mathbb{R}$, A is the $r \times r$ Brunovsky matrix:

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \in R^{r\times r}$$

$C = (0, \cdots, 0, 1) \in \mathbb{R}^{1\times r}$, $\beta : \mathbb{R} \to \mathbb{R}^n$ and $\eta : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$.

For the form (3), one can easily design a reduced order observer to estimate only the partial state ξ.

Lemma 1 *The following dynamical system:*

$$\left\{\begin{array}{l} \dot{\hat{\xi}} = A\hat{\xi} + \beta(y) + K(\hat{y} - y) \\ \hat{y} = C\hat{\xi} \end{array}\right.$$

is an observer for the proposed partial observability normal form (3).

Proof 1 *Set $e = \xi - \hat{\xi}$, we have $\dot{e} = (A - KC)\, e$.*

Since $A \in \mathbb{R}^{r\times r}$ is in the Brunovsky form and $C = (0, \cdots, 0, 1) \in \mathbb{R}^{1\times r}$, thus the pair (A, C) is observable. One can arbitrarily choose K such that $(A - KC)$ is Hurwitz, and this implies the exponential convergence of $\hat{\xi}$ to ξ.

It is shown that once system (1) can be transformed via a diffeomorphism $(\xi^T, \zeta^T)^T = \phi(x)$ into the partial observability normal form (3), then one can design the above simple observer to estimate ξ. Moreover, if z is a function such that:

$$\frac{\partial z}{\partial \zeta} = 0$$

then z is Z-observable for (1), and we can use ξ to estimate z.

Therefore, the rest of paper deals with the deduction of necessary and sufficient conditions which guarantee a diffeomorphism to transform system (1) into the proposed partial observability normal form (3).

Remark 1 *The partial observability normal form considered in this work is quite different from the normal form introduced in the work of Röbenack and Lynch [20]. which is written as:*

$$\left\{\begin{array}{l} \dot{\xi} = A\xi + \beta(y,\zeta) \\ \dot{\zeta} = \eta(\xi,\zeta) \\ y = C\xi \end{array}\right.$$

where β depends also on the second variable ζ.

4 Nonlinear systems without inputs

In this paper, it is assumed that there exists $r < n$ such that rank$\left\{dh, dL_f h, \cdots, dL_f^k h, \cdots\right\}$ is r. Thus system (1) is not fully observable. For $1 \le i \le r$, set $\theta_i = dL_f^{i-1}h$ and Δ =span$\{\theta_1, \theta_2, \cdots, \theta_r\}$. Denote $\Delta^\perp$ =ker Δ the distribution kernel of Δ.

Let τ_1 be a vector field modulo $\Delta^\perp$ which satisfies the following conditions:

$$\left\{\begin{array}{l} dL_f^k h(\tau_1) = 0 \text{ for } 0 \le k \le r-2 \\ dL_f^{r-1} h(\tau_1) = 1 \end{array}\right.$$

and by induction define the following family of vector fields $\tau_i = [\tau_{i-1}, f]$ modulo $\Delta^\perp$ for $2 \le i \le r$, which implies $\tau_i - [\tau_{i-1}, f] \in \Delta^\perp$, where $[,]$ denotes the conventional Lie bracket. Thus, one can choose a complementary family of vector fields $\{\tau_{r+1}, ..., \tau_n\}$ such that $\tau = [\tau_1, \tau_2, \cdots, \tau_n]$ forms a basis and $\theta_k(\tau_j) = 0$ for $1 \le k \le r$, $r+1 \le j \le n$.

Note

$$\Lambda_1 = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_r \end{pmatrix} (\tau_1, \tau_2, \cdots, \tau_r) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdots & 1 & * \\ 0 & \cdots & * & * \\ 1 & \cdots & * & * \end{pmatrix}$$

With the chosen $\{\tau_{r+1}, \tau_{r+2}, \cdots, \tau_n\}$, one can freely choose $\{\theta_{r+1}, \theta_{r+2}, \cdots, \theta_n\}$ such that

$$\Lambda_2 = \begin{pmatrix} \theta_{r+1} \\ \vdots \\ \theta_n \end{pmatrix} (\tau_{r+1}, \cdots, \tau_n)$$

is of rank $n - r$.

Property 1 *By giving the vector fields $(\tau_1, \tau_2, \cdots, \tau_r)$ and the codistribution $(\theta_1, \theta_2, \cdots, \theta_r)$, the chosen complementary τ_i and θ_i for $r+1 \le i \le n$ should satisfy the following properties*

1) $\tau = [\tau_1, \tau_2, \cdots, \tau_n]$ *forms a basis;*
2) $\theta_k(\tau_j) = 0$ *for* $1 \le k \le r$ *and* $r+1 \le j \le n$.
3) *rank*$\Lambda_2 = n - r$

Since τ is a basis, then it can be viewed as an invertible matrix. Therefore, in this basis f can be decomposed as follows:

$$f(x) = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i} = F_1 + F_2 \tag{4}$$

with $F_1 = \sum_{i=1}^{r} F_{1,i}(x)\tau_i$ and $F_2 = \sum_{j=r+1}^{n} F_{2,j}(x)\tau_j$.

One can state the following theorem.

Theorem 1 *Given a family of vector fields τ and θ satisfied Property 1, there exists a diffeomorphism $(\xi^T, \zeta^T)^T = \phi(x)$ which transforms the dynamical system (1) into the partial observability normal form (3) if and only if*

- $[\tau_i, \tau_j] = 0$ *for all* $1 \le i \le n$ *and* $1 \le j \le n$;
- $[\tau_i, F_1] = 0$ *for all* $r+1 \le i \le n$, *where* F_1 *is defined in (4).*

Proof 2 ***Necessity:***

If there existes a diffeomorphism $(\xi^T,\zeta^T)^T = \phi(x)$ *which transforms (1) into the form (3), then one has* $\phi_*(\tau_i) = \frac{\partial}{\partial \xi_i}$ *for* $1 \le i \le n$*, which implies* $[\phi_*(\tau_i),\phi_*(\tau_j)] = \phi_*([\tau_i,\tau_j]) = 0$ *for* $1 \le i \le n$*. Since* $\xi = \phi(x)$ *is a diffeomorphism, one has* $[\tau_i,\tau_j] = 0$ *for* $1 \le i \le n$ *and* $1 \le j \le n$*.*

Moreover, it is easy to see that for the dynamical system (3) we have $\Delta^\perp$ *=span*$\{\frac{\partial}{\partial \zeta_i},\ r+1 \le i \le n\}$*, therefore* $\phi_*(\tau_i) = \frac{\partial}{\partial \xi_i}$ *modulo* $\Delta^\perp$*. Thus it is easy to check that*

$$[\phi_*(\tau_i),\phi_*(F_1)] = \phi_*([\tau_i,F_1]) = 0$$

for $r+1 \le i \le n$*. Finally one obtains* $[\tau_i,F_1] = 0$ *and then we have*

$$\phi_*(f) = \begin{pmatrix} A\xi+\beta(y) \\ \eta(\xi,\zeta) \end{pmatrix}$$

which implies that

$$\phi_*(F_1)+\phi_*(F_2) = \begin{pmatrix} A\xi+\beta(y) \\ \eta(\xi,\zeta) \end{pmatrix}$$

Sufficiency*:*

Since $[\tau_i,\tau_j] = 0$ *for* $1 \le i \le n$ *and* $1 \le j \le n$*, then according to Poincaré's lemma there exists locally a diffeomorphism*

$$(\xi^T,\zeta^T)^T = \phi(x)$$

such that $d\phi = \phi_*$ *with* $\phi_*(\tau_i) = \frac{\partial}{\partial \xi_i}$ *for* $1 \le i \le r$ *and* $\phi_*(\tau_j) = \frac{\partial}{\partial \zeta_j}$ *for* $r+1 \le j \le n$*. Thus, for* $1 \le i \le r-1$*, one has*

$$\begin{aligned} \frac{\partial \phi_*(f)}{\partial \xi_i} &= \phi_*([\tau_i,f]) = \phi_*(\tau_{i+1} \textit{ modulo } \Delta^\perp) \\ &= \frac{\partial}{\partial \xi_{i+1}} \textit{ modulo span}\{\frac{\partial}{\partial \zeta_k}\ \ r+1 \le k \le n\} \end{aligned}$$

Moreover, for $r+1 \le j \le n$*, since* $[\tau_j,F_1] = 0$*, hence*

$$\frac{\partial \phi_*(F_1)}{\partial \zeta_j} = \phi_*([\tau_j,F_1]) = 0 \tag{5}$$

By integration one has

$$\dot{\xi} = \phi_*(F_1) = \sum_{i=1}^{r} \xi_i \frac{\partial}{\partial \xi_{i+1}} + \beta(y)$$

and this proved the sufficiency.

Note $\theta = (\theta_1,\cdots,\theta_n)^T$, one can define the following matrix:

$$\Lambda = \theta\tau = \begin{pmatrix} \Lambda_1 & 0_{r\times(n-r)} \\ * & \Lambda_2 \end{pmatrix}$$

It is clear that Λ is invertible, thus one can define the following multi 1-forms:

$$\omega = \Lambda^{-1}\theta \tag{6}$$

Theorem 2 *Suppose that Theorem 1 is fulfilled, then the diffeomorphism* $(\xi^T,\zeta^T)^T = \phi(x)$ *which transforms (1) into the form (3) is determined by*

$$\phi(x) = \int \omega + \phi(0)$$

Proof 3 *For any two vector fields* X, Y *one has*

$$d\omega(X,Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X,Y])$$

By setting $X = \tau_i$ *and* $Y = \tau_j$*, one obtains*

$$d\omega(\tau_i,\tau_j) = L_{\tau_i}\omega(\tau_j) - L_{\tau_j}\omega(\tau_i) - \omega([\tau_i,\tau_j])$$

As $\omega(\tau_j)$ *and* $\omega(\tau_i)$ *are constant, then one has*

$$d\omega(\tau_i,\tau_j) = -\omega([\tau_i,\tau_j])$$

which implies the equivalence between $d\omega = 0$ *and* $[\tau_i,\tau_j] = 0$*.*

Since Theorem 1 is fulfilled, thus one always has $d\omega = 0$*. According to Poincaré's lemma, this implies that there exists a diffeomorphism* $\phi(x)$ *such that* $\omega = d\phi$*. Finally the diffeomorphism can be determined just by integration of* ω *defined in (6).*

Example 1 *Let us consider the following dynamical system:*

$$\begin{cases} \dot{x}_1 = -x_3^2 + x_3x_2x_1 - \frac{1}{2}x_3^3 \\ \dot{x}_2 = x_1 - \frac{1}{2}x_3^2 \\ \dot{x}_3 = -x_3 + x_2x_1 - \frac{1}{2}x_3^2 \\ y = x_2 \\ z = x_2 + 2x_1x_2 - x_2x_3^2 \end{cases} \tag{7}$$

A simple calculation gives rank$\left\{dh, dL_fh, dL_f^2h\right\} = 2$*, thus* $r = 2$*. One has*

$$\theta_1 = dx_2 \ \text{ and } \theta_2 = dx_1 - x_3dx_3$$

Let Δ *=span*$\{\theta_1,\theta_2\}$*, and*

$$\ker\Delta = \textit{span}\left\{l_1 = x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right\}$$

The frame τ *is given by*

$$\begin{aligned} \tau_1 &= \frac{\partial}{\partial x_1} \\ \tau_2 &= \frac{\partial}{\partial x_2} + x_2x_3\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} \textit{ modulo } l_1 = \frac{\partial}{\partial x_2} \end{aligned}$$

In order to form a basis which satisfies $\theta_1(\tau_3) = \theta_2(\tau_3) = 0$*, the third complementary vector field can be chosen as follows*

$$\tau_3 = \frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_1}$$

which makes the following equality be satisfied

$$[\tau_1,\tau_2] = [\tau_1,\tau_3] = [\tau_2,\tau_3] = 0$$

According to 4), one has

$$f = F_1 + F_2 = F_{12}\tau_2 + F_{23}\tau_3$$

where

$$F_1 = \left(x_1 - x_3^2/2\right)\tau_2$$

and $F_2 = \left(-x_3 + x_1x_2 - x_3^2/2\right)\tau_3$*. It can be checked that*

$$[\tau_3, F_1] = 0$$

Then the second item of Theorem 1 is satisfied.

In order to have rank Λ_2 *=rank* $\{\theta_3(\tau_3)\} = 1$*, one can choose*

$$\theta_3 = dx_3 - x_2 dx_2$$

thus

$$\Lambda = \theta\tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which yields

$$\omega = \Lambda^{-1}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} d\xi_1 \\ d\xi_2 \\ d\zeta_1 \end{pmatrix} = d\begin{pmatrix} x_1 - \frac{1}{2}x_3^2 \\ x_2 \\ x_3 \end{pmatrix}$$

Finally, one obtains the following diffeomorphism

$$\phi(x) = \begin{pmatrix} x_1 - \frac{1}{2}x_3^2 \\ x_2 \\ x_3 \end{pmatrix}$$

which transforms the studied system into the following form

$$\begin{cases} \dot{\xi}_1 = 0 \\ \dot{\xi}_2 = \xi_1 \\ \dot{\zeta}_1 = \eta(\xi_1, \xi_2, \zeta) \\ y = \xi_2 \end{cases}$$

Moreover, one has

$$z = x_2 + 2x_1x_2 - x_2x_3^2 = \xi_2 + 2\xi_1\xi_2$$

and $\frac{\partial z}{\partial \zeta_1} = 0$ *which implies that* z *is* Z*-observable, and one can use the estimated* ξ *to recover* z *in (7).*

5 Extension to systems with inputs

In this section, we extend our results to systems with inputs in the following form:

$$\begin{cases} \dot{x} = f(x) + \sum_{k=1}^{m} g_k(x) u_k \quad (2) \\ y = h(x) \end{cases} \tag{8}$$

where $x \in \mathbb{R}^n$ is the state, $u = (u_1, ..., u_m)^T \in \mathbb{R}^m$ is the inputs, $y \in \mathbb{R}$ is the output, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g_k : \mathbb{R}^n \to \mathbb{R}^n$and $h : \mathbb{R}^n \to \mathbb{R}$ are sufficiently smooth. For system (8), the partial observability normal form is as follows:

$$\begin{cases} \dot{\xi} = A\xi + \beta(y) + \sum_{k=1}^{m} \alpha_k^1(y) u_k \\ \dot{\zeta} = \eta(\xi, \zeta) + \sum_{k=1}^{m} \alpha_k^2(\xi, \zeta) u_k \\ y = C\xi \end{cases} \tag{9}$$

where A, C, β and η are the same as those defined in the form (3).

Following the same procedure, let define the projection of g_k on τ as follows:

$$G_k = G_k^1 + G_k^2$$

with

$G_k^1 = \sum_{i=1}^{r} G_k^{1,i}(x)\,\tau_i$ and $G_k^2 = \sum_{j=r+1}^{n} G_k^{2,j}(x)\,\tau_j$.

Then we have the following theorem.

Theorem 3 *Suppose that Theorem 1 is satisfied. There exists a diffeomorphism*$(\xi^T, \zeta^T)^T = \phi(x)$ *which transforms (8) into the form (9) if and only if*

$$\left[\tau_i, G_k^1\right] = 0$$

for $1 \le i \le n, i \ne r$ *and* $1 \le k \le m$*.*

Proof 4 *From Theorem 1, one can state that there exists a diffeomorphism such that*

$$\phi_*(F_1) = A(y)z + \beta(y)$$

Now, for $r + 1 \le i \le n$ *and* $1 \le k \le m$*, one has*

$$\frac{\partial \phi_*\left(G_k^1\right)}{\partial \zeta_i} = \left[\frac{\partial}{\partial \zeta_i}, \phi_*\left(G_k^1\right)\right] = \phi_*\left[\tau_i, G_k^1\right] = 0$$

It is the same for $1 \le i < r$ *such that* $\phi_*\left[\tau_i, G_k^1\right] = 0$ *Therefore* $\phi_*(G_k^1) = \alpha_k^1(y)$*, and finally we proved Theorem 3.*

6 Diffeomorphism on the output

By giving a family of vector fields τ and θ satisfied Property 1, if the conditions of Theorem 1 cannot be fulfilled, i.e. the Lie brackets of vector fields do not commute, then one can modify those vector fields to construct a new family of commutative vector fields, by applying another diffeomorphism on the output (see [3, 11]).

For this, let τ_1 be the vector field modulo $\Delta^\perp$ defined in Section 4. Denote $s(y) \ne 0$ a function of the output of (1), and one can construct a new vector field σ_1 according to the following equation:

$$\sigma_1 = s(y)\tau_1$$

and by induction define the following new family of vector fields

$$\sigma_k = [\sigma_{k-1}, f] \text{ modulo } \Delta^\perp \text{ for } 2 \le k \le r$$

Thus, one can choose a complementary family of vector fields $\{\sigma_{r+1}, ..., \sigma_n\}$ such that $\sigma = [\sigma_1, \sigma_2, \cdots, \sigma_n]$ forms a basis and $\theta_k(\sigma_j) = 0$ for $1 \le k \le r$, $r + 1 \le j \le n$.

Note

$$\tilde{\Lambda}_1 = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_r \end{pmatrix}(\sigma_1, \sigma_2, \cdots, \sigma_r) = \begin{pmatrix} 0 & \cdots & 0 & s \\ \vdots & \cdots & s & * \\ 0 & \cdots & * & * \\ s & \cdots & * & * \end{pmatrix}$$

With the chosen $\{\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n\}$, one can freely choose $\{\theta_{r+1}, \theta_{r+2}, \cdots, \theta_n\}$ such that

$$\tilde{\Lambda}_2 = \begin{pmatrix} \theta_{r+1} \\ \theta_{r+2} \\ \vdots \\ \theta_n \end{pmatrix}(\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n)$$

is of rank $n - r$.

Property 2 *By giving the vector fields* $(\sigma_1, \sigma_2, \cdots, \sigma_r)$ *and the codistribution* $(\theta_1, \theta_2, \cdots, \theta_r)$*, the chosen complementary* σ_i *and* θ_i *for* $r + 1 \le i \le n$ *should satisfy the following properties*

1) $\sigma = [\sigma_1, \tau_2, \cdots, \sigma_n]$ *forms a basis;*
2) $\theta_k(\sigma_j) = 0$ *for* $1 \leq k \leq r$ *and* $r+1 \leq j \leq n$.
3) *rank*$\tilde{\Lambda}_2 = n - r$

Then, based on the new basis σ, f can be decomposed as follows:

$$f = \sum_{i=1}^{r} F_{1,i}(x)\sigma_i + \sum_{j=r+1}^{n} F_{2,j}(x)\sigma_j \tag{10}$$

with $F_1 = \sum_{i=1}^{r} F_{1,i}(x)\sigma_i$ and $F_2 = \sum_{j=r+1}^{n} F_{2,j}(x)\sigma_j$. And one can state the following theorem.

Theorem 4 *Given an output function $s(y) \neq 0$ which construct a new family of vector fields σ and θ satisfied Property 2, there exists a diffeomorphism $(\xi^T, \zeta^T)^T = \phi(x)$ which transforms the dynamical system (1) into the partial observability normal form (3) with $\xi_r = \bar{y} = \psi(y)$ where $\psi(y) = \int_0^y \frac{1}{s(c)} dc$. if and only if*

- $[\sigma_i, \sigma_j] = 0$ *for all* $1 \leq i \leq n$ *and* $1 \leq j \leq n$;
- $[\sigma_i, F_1] = 0$ *for all* $r+1 \leq i \leq n$, *where* F_1 *is defined in (10).*

Proof 5 *The proof of this theorem is similar with that of Theorem 1, thus is omitted.*

Remark 2 *The deduction of such an output function $s(y) \neq 0$ in Theorem 4 is exhaustively investigated in [11].*

Remark 3 *Following the same arguments in Section 4, the diffeomorphism $\phi(x)$ can be calculated by using $\phi(x) = \int \tilde{\omega} + \phi(0)$ where $\tilde{\omega} = \tilde{\Lambda}^{-1}\theta$ with $\tilde{\Lambda} = \theta\sigma$.*

The following example highlights the proposed result.

Example 2 *Let consider the well-known SIR epidemic model that undergoes the spread of a contagious disease as follows:*

$$\begin{cases} \dot{S} = -\beta S I \\ \dot{I} = \beta S I - \gamma I \\ \dot{R} = \gamma I \\ y = I \\ z = l(S, I) = N - I - S \end{cases}$$

where S denotes the suspected population, I denotes the infected, R denotes the removed population and the total population N is assumed to be known. The objective is to apply the proposed result of this paper to estimate the function $l(S, I) = N - I - S$.

By using the same notations as in Section 4, a simple calculation gives:

$$\theta_1 = dI \;\; \text{and} \; \theta_2 = \beta I dS + (\beta S - \gamma)\, dI$$

which yield the following vector fields:

$$\tau_1 = \frac{1}{\beta I}\frac{\partial}{\partial S} \;\; \text{and} \; \tau_2 = \frac{\partial}{\partial I} + (\beta S - \gamma - \beta I)\, \tau_1$$

Unfortunately, these two vector fields do not commute, since $[\tau_1, \tau_2] = \frac{2}{I}\tau_1$. In order to construct a new family of commutative vector fields by introducing a diffeomorphism on the output, let follow the method proposed in [11] to deduce a non-zero output function $s(y)$. For this, set $\sigma_1 = s(y)\tau_1$ and $\sigma_2 := [\sigma_1, f]$ modulo $\Delta^{\perp} = s(y)\tau_2 - s'(y)\,(\beta S I - \gamma I)\,\tau_1$. Now, a straightforward calculation gives: $[\sigma_1, \sigma_2] = (\frac{2s^2(y)}{I} - 2s(y)s^{'}(y))\tau_1$, thus $[\sigma_1, \sigma_2] = 0$ if and only if function $s(y)$ fulfils the following differential equation:

$$\frac{s(y)}{y} - \frac{ds(y)}{dy} = 0$$

Thus one can choose $s(y) = y = I$ which yields

$$\sigma_1 = \frac{1}{\beta}\frac{\partial}{\partial S} \;\; \text{and} \;\; \sigma_2 = -I\frac{\partial}{\partial S} + I\frac{\partial}{\partial I}$$

In order to construct σ and θ satisfying Property 2, one chooses $\sigma_3 = \frac{\partial}{\partial R}$ and $\theta_3 = dR$ which makes

$$[\sigma_1, \sigma_2] = [\sigma_1, \sigma_3] = [\sigma_2, \sigma_3] = 0$$

Based on the new basis σ, according to (10), one obtains

$$f = -\beta\gamma I \tau_1 + (\beta S - \gamma)\, \tau_2 + \gamma I \tau_3$$

then $F_1 = -\beta\gamma I\tau_1 + (\beta S - \gamma)\,\tau_1$. It can be checked that

$$[\tau_3, F_1] = 0$$

and the second item of Theorem 4 is satisfied.

Since

$$\tilde{\Lambda} = \theta\sigma = \begin{pmatrix} 0 & I & 0 \\ I & -\beta I^2 + (-\gamma + S\beta)\, I & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which yields

$$\tilde{\omega} = \tilde{\Lambda}^{-1}\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} d\xi_1 \\ d\xi_2 \\ d\zeta_1 \end{pmatrix} = d\begin{pmatrix} \beta(S+I) \\ \ln I \\ R \end{pmatrix}$$

Therefore, the diffeomorphism is given as follows:

$$\phi\,(x) = \begin{pmatrix} \beta(S+I) \\ \ln I \\ R \end{pmatrix}$$

which transforms the studied system into the following form

$$\begin{cases} \dot{\xi}_1 = -\beta\gamma e^{\bar{y}} \\ \dot{\xi}_2 = \xi_1 - \beta e^{\bar{y}} - \gamma \\ \dot{\zeta}_1 = \gamma e^{\bar{y}} \\ \bar{y} = \xi_2 = \ln I \end{cases}$$

In the transformed form, the Z-function becomes

$$z = l(S, I) = \tilde{l}\,(\xi) = N - \frac{\xi_1}{\beta}$$

which is independent of ζ_1, thus it is Z-observable, and one can use the estimated ξ_1 to recover $l(S, I)$.

By setting $\beta = 0.001$, $\gamma = 0.1$, the simulation results are depicted in Fig. 1-2 for the estimation of suspected and infected population.

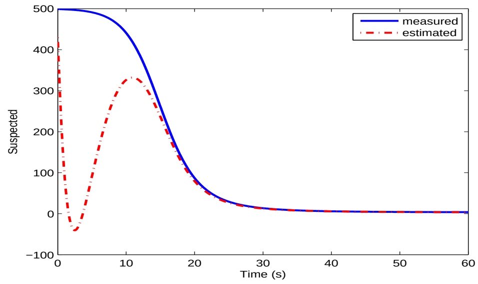

Fig. 1: Estimation of suspected population (S)

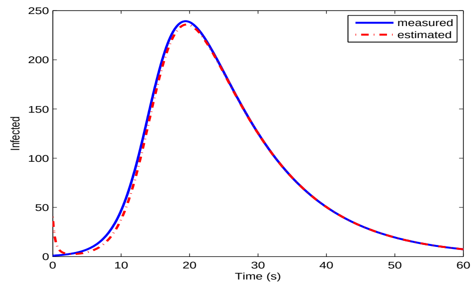

Fig. 2: Estimation of infected population (I)

7 Conclusion

This paper deals with partial observability problem, which is useful sometimes when for example only partial states are need to be estimated, or only functions of certain states are required to be estimated. For this, a partial observability normal form is presented, for which a simple Luenberger observer can be applied directly to estimate the partial states. Necessary and sufficient conditions are deduced which guarantee the existence of a diffeomorphism transforming nonlinear systems into the proposed partial observability normal form. The results are extended to nonlinear systems with inputs and the transformation on the output as well. A concrete example was provided to illustrate the feasibility of the proposed results.

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